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## NOTES AND CORRESPONDENCE

## A Proof for the Equivalence of Two Upper Bounds for the Growth of Disturbances from Barotropic Instability

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### Abstract

A previous study proposed two methods for calculating the upper bound of the growth of disturbances from barotropic instability of a zonal flow in a two-dimensional incompressible fluid on a rotating sphere. The study conjectured that these two upper bounds are equivalent. One method was based on the conservation of the domain-averaged pseudomomentum density, and the other solved a minimization problem under the constraints of the conservations of all Casimir invariants and the total absolute angular momentum. In this study, this conjecture is verified, i.e., a proof is presented for their equivalence by developing an annealing-like procedure to reach the absolute vorticity profile that corresponds to the upper bound. The procedure also provides a more efficient method to calculate the upper bound.

**Keywords** barotropic instability; nonlinear stability; pseudomomentum; upper bound for disturbance growth; optimization problem

### 1. Introduction

If a flow is unstable, disturbances grow; however, their growth does not continue indefinitely. There exists some upper bounds to the growth. Shepherd (1988) proposed a method to calculate a fully nonlinear rigorous upper bound for the growth of disturbances from barotropic instability using the conservation of pseudomomentum density based on the nonlinear stability theorem given by Arnol'd (1966).

The upper bound, however, was not the tightest bound under the constraints of the conservation of all considered invariants. A tighter bound was obtained by Ishioka and Yoden (1996) by revising the method of Shepherd (1988). Ishioka and Yoden (1996) also proposed a new method to calculate the tightest bound

under the constraints of the conservation of all considered invariants. They applied these two methods, namely, the revised version of the method of Shepherd (1988) and the method of calculating the tightest bound, to several basic flow profiles and showed that the values of the two upper bounds calculated using these two methods were approximately equivalent, with a relative error of ~1%. This implies that the revised version of the method of Shepherd (1988) can yield the tightest bound under the considered constraints. No proof for the equivalence, however, has yet been reported.

In this paper, a proof for the equivalence is presented, and a more efficient method is proposed to calculate the upper bound. The remainder of the paper is organized as follows. The problem is described in Section 2. The two methods proposed by Ishioka and Yoden (1996) for calculating the upper bounds are introduced in Section 3. In Section 4, a proof for the equivalence of the two bounds is presented, and an efficient new method for calculating the bound is

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proposed. Finally, a discussion and summary are presented in Section 5.

## 2. Problem setup

The system under consideration is an incompressible, inviscid, two-dimensional fluid flowing over a rotating sphere. The governing equation is the following vorticity equation, which is nondimensionalized by the radius and the rate of rotation of the sphere:

$$\frac{\partial q}{\partial t} + \left( \frac{\partial \psi}{\partial \lambda} \frac{\partial q}{\partial \mu} - \frac{\partial \psi}{\partial \mu} \frac{\partial q}{\partial \lambda} \right) = 0. \quad (1)$$

Here,  $q \equiv \nabla^2 \psi + 2\mu$  is the absolute vorticity,  $\psi(\lambda, \mu, t)$  is the stream function,  $t$  is the time,  $\lambda$  is the longitude,  $\mu$  is the sine latitude, and  $\nabla^2$  is the Laplacian on a unit sphere, which is defined as

$$\nabla^2 = \frac{1}{1-\mu^2} \frac{\partial^2}{\partial \lambda^2} + \frac{\partial}{\partial \mu} \left\{ (1-\mu^2) \frac{\partial}{\partial \mu} \right\}.$$

The absolute vorticity,  $q$ , is conserved, following the motion of the fluid governed by (1). Since the fluid motion is also incompressible, the quantity  $C_f$ , which is defined as

$$C_f = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 f(q) d\mu d\lambda,$$

is an invariant. Here  $f(q)$  is an arbitrary function of  $q$ . If we set  $f(q) = \frac{1}{2}q^2$ ,  $C_f$  becomes the total enstrophy  $F$ . The other invariants, the total angular momentum  $D$  and total energy  $E$ , are defined as follows:

$$D = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 \mu q d\mu d\lambda,$$

$$E = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 -\frac{1}{2} \psi \nabla^2 \psi d\mu d\lambda.$$

The conservation of  $E$ , however, is not used in this paper.

Now, let us consider the time evolution of (1) from the following initial condition:

$$q = q_{\text{initial}}(\mu) + (\text{an infinitesimal disturbance}),$$

where  $q_{\text{initial}}(\mu)$  is a nonmonotonic function of  $\mu$ , so that barotropic instability can occur. Defining  $q$  as  $q = \bar{q} + \tilde{q}$ , we obtain the zonal enstrophy and wave enstrophy as follows:

$$F_z = \frac{1}{2} \int_{-1}^1 \frac{1}{2} (\bar{q})^2 d\mu, \quad F_w = \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 \frac{1}{2} (\tilde{q})^2 d\mu d\lambda;$$

$$F = F_z + F_w,$$

where  $(\bar{\quad})$  denotes the zonal mean, which is defined as

$$\bar{q} = \frac{1}{2\pi} \int_0^{2\pi} q d\lambda.$$

Under the above settings, we consider how to obtain the upper bound of  $F_w$ , or equivalently, the lower bound of  $F_z$ , under the constraints of the conservation of any  $C_f$  and  $D$ . In the next section, we introduce the two methods developed by Ishioka and Yoden (1996) to calculate the upper bound.

## 3. Outline of the two upper bounds

### 3.1 The direct bound

To conserve all  $C_f$ s, any possible distribution of  $q$  must be a rearrangement from the initial  $q$  distribution. Since we intend to minimize  $F_z$ , which is the zonal component of  $F$ , we divide the sphere into  $M$  latitudinal belts of equal area. Numbering the belts starting from the south, we define the  $j$ -th belt to occupy the interval  $-1 + (j-1)\Delta\mu \leq \mu \leq -1 + j\Delta\mu$  in the  $\mu$ -coordinate ( $j = 1, 2, \dots, M$ ). Here  $\Delta\mu = 2/M$ . Normalized by the total area of the sphere, each latitudinal belt has an area of  $1/M$ . Whereas, Ishioka and Yoden (1996) divided the sphere unequally, we adopt the equal division as described above for convenience in the proof in the next section. For each latitudinal belt, we define  $\mu_j$  as

$$\mu_j = -1 + (j-1/2)\Delta\mu \quad (j = 1, 2, \dots, M),$$

which we regard as the representative  $\mu$  value of the  $j$ -th belt.

If we introduce  $r_{ij}$  ( $i, j = 1, 2, \dots, M$ ) to indicate the area of the air parcel that is initially in the  $i$ -th belt and then moves to the  $j$ -th belt, we can describe any rearrangement of air parcels using the matrix  $(r_{ij})$ . Considering that an area cannot be negative and each belt has an area of  $1/M$ , the constraints that must be satisfied by  $(r_{ij})$  are written as follows:

$$r_{ij} \geq 0 \quad (i = 1, 2, \dots, M; j = 1, 2, \dots, M), \quad (2)$$

$$\sum_{i=1}^M r_{ij} = 1/M \quad (j = 1, 2, \dots, M), \quad (3)$$

$$\sum_{j=1}^M r_{ij} = 1/M \quad (i = 1, 2, \dots, M). \quad (4)$$

Hereafter, we refer to any rearrangement of air parcels described using  $(r_{ij})$  that satisfies constraints (2) through (4) as air parcel exchange.

Assuming the absolute vorticity of the air parcel that is initially in the  $i$ -th belt as  $q_i = q_{\text{initial}}(\mu_i)$  ( $i = 1, 2, \dots, M$ ), the quantity  $C_f$  for the discretized system is defined as follows:

$$C_f = \sum_{j=1}^M \sum_{i=1}^M f(q_i) r_{ij} = \frac{1}{M} \sum_{i=1}^M f(q_i), \quad (5)$$

where constraint (4) has been used in the last equality. Since  $C_f$  does not depend on  $(r_{ij})$ ,  $C_f$  is also conserved in the discretized system as a matter of course. When  $f(q) = \frac{1}{2}q^2$ ,  $C_f$  becomes the total enstrophy  $F$ , which is written as

$$F = \frac{1}{M} \sum_{i=1}^M \frac{1}{2} (q_i)^2. \quad (6)$$

The average absolute vorticity in the  $j$ -th belt can be defined as

$$\bar{q}_j = M \sum_{i=1}^M q_i r_{ij} \quad (j = 1, 2, \dots, M). \quad (7)$$

Using  $\bar{q}_j$  ( $j = 1, 2, \dots, M$ ), the total angular momentum  $D$  and the zonal enstrophy  $F_z$  can be defined as follows:

$$D = \frac{1}{M} \sum_{j=1}^M \mu_j \bar{q}_j, \quad (8)$$

$$F_z = \frac{1}{M} \sum_{j=1}^M \frac{1}{2} (\bar{q}_j)^2. \quad (9)$$

The wave enstrophy  $F_w$  is written as

$$\begin{aligned} F_w &= F - F_z = \sum_{j=1}^M \sum_{i=1}^M \frac{1}{2} q_i^2 r_{ij} - \frac{1}{M} \sum_{j=1}^M \frac{1}{2} (\bar{q}_j)^2 \\ &= \sum_{j=1}^M \sum_{i=1}^M \frac{1}{2} (q_i - \bar{q}_j)^2 r_{ij}. \end{aligned} \quad (10)$$

Now, we consider a minimization problem for  $F_z$  under constraints (2) through (4) for  $(r_{ij})$  and the conservation of  $D$ . Here, since the initial value of  $D$  before air parcel exchange can be written as

$$D_{\text{initial}} = \frac{1}{M} \sum_{j=1}^M \mu_j q_j,$$

the conservation of  $D$  is represented as follows:

$$D = D_{\text{initial}}. \quad (11)$$

Finally, we have formulated the procedure to seek the upper bound of  $F_w$ , which is defined by (10), through the minimization for  $F_z$  under constraints (2) through (4) for  $(r_{ij})$  and constraint (11), which is also considered to be a constraint for  $(r_{ij})$  through (7) and (8). This problem is a convex quadratic programming problem. The upper bound of  $F_w$ , which can be computed by solving the convex quadratic programming problem, is referred to as the direct bound by Ishioka and Yoden (1996).

### 3.2 The revised Shepherd's bound

In this subsection, we explain the revised Shepherd's bound, which was introduced by Ishioka and Yoden (1996). For convenience in the proof presented in the next section, the explanation here is based on the discretized system described in the previous subsection.

After defining  $q_{\min}$  and  $q_{\max}$  as the minimum and the maximum values of  $q_i$  ( $i = 1, 2, \dots, M$ ), respectively, we introduce  $Y(\eta)$  as a nondecreasing and piecewise differentiable (may be discontinuous) function of  $\eta$ , for which the domain of definition is  $q_{\min} \leq \eta \leq q_{\max}$ . As seen later,  $Y(\eta)$  is the inverse of the basic absolute vorticity profile whose nonlinear stability is considered. Here, we introduce a nondecreasing  $Y(\eta)$  because we are assuming that  $D_{\text{initial}}$  is positive. If  $D_{\text{initial}}$  was negative, we would introduce a nonincreasing  $Y(\eta)$ . Next we define  $Q(\mu)$  as the inverse of  $Y(\eta)$ . Note that if  $Y(\eta)$  jumps from  $Y_{d1}$  to  $Y_{d2}$  at a discontinuous point  $\eta_d$ , we define  $Q(\mu)$  in this interval as  $Q(\mu) = \eta_d$  ( $Y_{d1} \leq \mu \leq Y_{d2}$ ). To ensure that the domain of the definition of  $Q(\mu)$  includes  $[\mu_1, \mu_M]$ , we set  $Q(\mu) = q_{\min}$  ( $\mu \leq Y(q_{\min})$ ) if  $Y(q_{\min}) > \mu_1$  and  $Q(\mu) = q_{\max}$  ( $\mu \geq Y(q_{\max})$ ) if  $Y(q_{\max}) < \mu_M$ .

Using  $Y(\eta)$  and  $Q(\mu)$ , we define the following function:

$$A_Q(\mu, q) = - \int_{Q(\mu)}^q \{Y(\eta) - \mu\} d\eta,$$

where the domain of the definition is  $\mu_1 \leq \mu \leq \mu_M$  and  $q_{\min} \leq q \leq q_{\max}$ . This function corresponds to the pseudomomentum density if we consider  $Q(\mu)$  to be the basic state and  $q - Q$  to be the perturbation. Since  $Y(\eta)$  is nondecreasing,  $A_Q(\mu, q) \leq 0$ . We consider the following summation based on the function  $A_Q(\mu, q)$  and the matrix  $(r_{ij})$ :

$$\begin{aligned} & \sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, q_i) r_{ij} \\ &= \sum_{j=1}^M \sum_{i=1}^M \left( - \int_{Q(\mu_j)}^{q_i} \{Y(\eta) - \mu_j\} d\eta \right) r_{ij} \\ &= \sum_{j=1}^M \sum_{i=1}^M \left( - \int_{Q(\mu_j)}^{q_i} Y(\eta) d\eta + \mu_j (q_i - Q(\mu_j)) \right) r_{ij} \\ &= \sum_{j=1}^M \sum_{i=1}^M \left( - \int_{q_c}^{q_i} Y(\eta) d\eta + \int_{q_c}^{Q(\mu_j)} Y(\eta) d\eta \right. \\ & \quad \left. + \mu_j (q_i - Q(\mu_j)) \right) r_{ij} \\ &= - \frac{1}{M} \sum_{i=1}^M \int_{q_c}^{q_i} Y(\eta) d\eta + \frac{1}{M} \sum_{j=1}^M \int_{q_c}^{Q(\mu_j)} Y(\eta) d\eta + D \\ & \quad - \frac{1}{M} \sum_{j=1}^M \mu_j Q(\mu_j). \end{aligned} \quad (12)$$

Here constraints (3) and (4) have been used, and  $q_c$  is an arbitrary constant that satisfies  $q_{\min} \leq q_c \leq q_{\max}$ . Since only  $D$  depends on  $(r_{ij})$  in the rightmost side of (12) as

$$D = \frac{1}{M} \sum_{j=1}^M \mu_j \bar{q}_j = \sum_{j=1}^M \sum_{i=1}^M \mu_j q_i r_{ij},$$

the quantity  $\sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, q_i) r_{ij}$  becomes an invariant for air parcel exchange that conserves  $D$ . The value of this invariant is calculated as

$$\sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, q_i) r_{ij} = \frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, q_j)$$

since

$$r_{ij} = \frac{1}{M} \delta_{ij} \quad (i = 1, 2, \dots, M; j = 1, 2, \dots, M)$$

for the initial state before the air parcel exchange. Here  $\delta_{ij}$  is the Kronecker's delta. Furthermore, by choosing  $\bar{q}$  and  $\check{q}$  to be arbitrary real numbers that satisfy  $\bar{q} \in [q_{\min}, q_{\max}]$  and  $\bar{q} + \check{q} \in [q_{\min}, q_{\max}]$ , we obtain

$$\begin{aligned} -A_Q(\mu, \bar{q} + \check{q}) &= \int_{Q(\mu)}^{\bar{q} + \check{q}} \{Y(\eta) - \mu\} d\eta \\ &\geq \int_{Q(\mu)}^{\bar{q}} \{Y(\eta) - \mu\} d\eta + \{Y(\bar{q}) - \mu\} \check{q} + \frac{1}{2} (\check{q})^2 Y'_{\min} \end{aligned} \quad (13)$$

through the mean-value theorem. Here,  $Y'_{\min}$  is the minimum of  $dY/d\eta$  in the domain of the definition of  $Y(\eta)$ . Note that the inequality (13) holds even when  $Y(\eta)$  has discontinuous points. From the inequality (13), we obtain

$$\begin{aligned} &-\sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, q_i) r_{ij} \\ &= -\sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, \bar{q}_i + (q_i - \bar{q}_i)) r_{ij} \\ &\geq -\sum_{j=1}^M \sum_{i=1}^M A_Q(\mu_j, \bar{q}_i) r_{ij} + \sum_{j=1}^M \sum_{i=1}^M (Y(\bar{q}_i) - \mu_j) (q_i - \bar{q}_i) r_{ij} \\ &\quad + Y'_{\min} \sum_{j=1}^M \sum_{i=1}^M \frac{1}{2} (q_i - \bar{q}_i)^2 r_{ij} \\ &= -\frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, \bar{q}_j) + Y'_{\min} F_w. \end{aligned} \quad (14)$$

Since the leftmost side of (14) is an invariant, we obtain

$$-\frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, \bar{q}_j) + Y'_{\min} F_w \leq -\frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, q_j).$$

Furthermore, by considering

$$-\frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, \bar{q}_j) \geq 0,$$

we finally obtain

$$F_w \leq -\frac{1}{Y'_{\min}} \frac{1}{M} \sum_{j=1}^M A_Q(\mu_j, q_j). \quad (15)$$

If we can solve a variational problem of the function  $Y(\eta)$  to minimize the right-hand side of (15), we can

obtain an upper bound of  $F_w$ . This is the revised Shepherd's bound introduced by Ishioka and Yoden (1996).

#### 4. Proof for the equivalence of the two upper bounds

In this section, we prove that the two upper bounds introduced in the previous section, direct bound and revised Shepherd's bound, are equivalent. We demonstrate the proof by constructing the profile of absolute vorticity that corresponds to the two upper bounds, as shown in the following subsections.

##### 4.1 Preparation

First, we sort the initial profile  $q_i$  ( $i = 1, 2, \dots, M$ ) through air parcel exchange so that it becomes nondecreasing with respect to the suffix  $i$ . Let the air parcel exchange corresponding to this sorting be represented by

$$r_{ij}^{(0)} \quad (i = 1, 2, \dots, M; j = 1, 2, \dots, M).$$

Then, by assuming that the air parcel in the  $i$ -th belt is moved to the  $k_i$ -th belt by sorting, we can write

$$r_{ij}^{(0)} = \frac{1}{M} \delta_{ik_i} \quad (i = 1, 2, \dots, M; j = 1, 2, \dots, M).$$

By writing the profile of absolute vorticity obtained through sorting as

$$\bar{q}_j^{(0)} = \frac{1}{M} \sum_{i=1}^M q_i r_{ij}^{(0)} \quad (j = 1, 2, \dots, M),$$

the following inequality naturally holds:

$$\bar{q}_1^{(0)} \leq \bar{q}_2^{(0)} \leq \dots \leq \bar{q}_M^{(0)}.$$

If we write the total angular momentum corresponding to the sorted profile as

$$D^{(0)} = \frac{1}{M} \sum_{j=1}^M \mu_j \bar{q}_j^{(0)},$$

then the maximum  $D$  for any rearrangement of the distribution of absolute vorticity from the initial distribution is obtained. Therefore,  $D^{(0)} \geq D_{\text{initial}}$ .

Next, we consider the following process. Without considering the conservation of  $D$ , we change the profile of  $\bar{q}_j^{(0)}$  ( $j = 1, 2, \dots, M$ ) through air parcel exchange that satisfies constraints (2) through (4). Hereafter, we write  $\bar{q}_j^{(0)}$  ( $j = 1, 2, \dots, M$ ) as an  $M$ -dimensional vector,  $\bar{\mathbf{q}}^{(0)}$ .

For the given  $\bar{\mathbf{q}}^{(0)}$ , we draw a polyline connecting  $M$  points  $(\mu_j, \bar{q}_j^{(0)})$  ( $j = 1, 2, \dots, M$ ). The gradient of each segment of the polyline is written as

$$\frac{\bar{q}_{i+1}^{(0)} - \bar{q}_i^{(0)}}{\Delta\mu} \quad (j = 1, 2, \dots, M-1).$$

We write the maximum gradient as  $\beta^{(0)}$  and define the steepest interval as each sequence of segments that have the maximum gradient. If there are several steepest intervals, by definition, there must be one or more segments that have smaller gradients between the steepest intervals. We define the midpoint of the steepest interval as the midpoint of the sequence of segments that make up the interval.

We then calculate the following three types of gradient:

- (i) The next steepest gradient of segments.
- (ii) If one segment is placed between two steepest intervals (not including the next steepest found in (i)), we calculate the gradient of the line connecting the midpoints of these two steepest intervals. If such a segment is not unique, we choose the largest value of the gradients defined above.
- (iii) We choose the largest value of the gradients of the lines, each of which connects the midpoint of the steepest interval and the opposite endpoint of each segment adjacent to the interval.

We denote the largest value of the gradients obtained by (i) through (iii) above as  $\beta^{(1)}$ . For every steepest interval in the profile  $\bar{q}^{(0)}$ , we reduce the gradient to  $\beta^{(1)}$  by moving the points  $(\mu_j, \bar{q}_j^{(0)})$  in the interval with the midpoint fixed, which yields a new profile. Determining  $\beta^{(1)}$  and updating the profile are illustrated in Fig. 1. Representing the new profile as  $\bar{q}^{(1)}$ , the gradient of the steepest interval for this profile becomes  $\beta^{(1)}$ . This profile is reachable through air parcel exchange from the profile of  $\bar{q}^{(0)}$ .

As described above, we have developed a process to obtain a new profile  $\bar{q}^{(1)}$ , whose maximum gradient is  $\beta^{(1)} (< \beta^{(0)})$ , through air parcel exchange from the starting profile  $\bar{q}^{(0)}$ , whose maximum gradient is  $\beta^{(0)}$ . Here air parcel exchange occurs only in the belts corresponding to the nodes included in the steepest intervals of the new profile (including the endpoint nodes of the intervals).

From the obtained profile  $\bar{q}^{(1)}$ , we can repeat the same process and update the profile in succession, which monotonically decreases the value of the maximum gradient. Corresponding to the repetition, the total angular momentum  $D$  decreases monotonical-

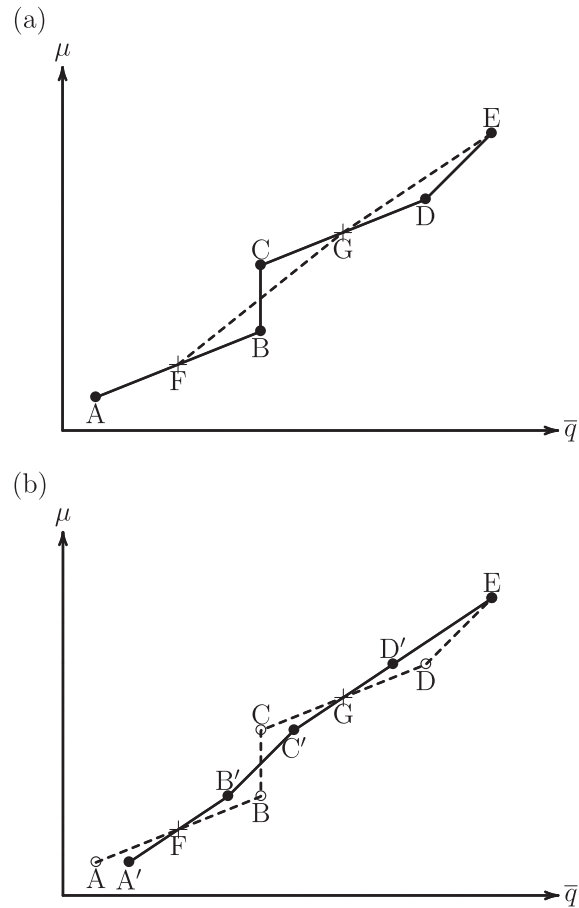


Fig. 1. Conceptual figures indicating the determination of  $\beta^{(1)}$  and the updating of the profile. (a): Determination of  $\beta^{(1)}$ . The black dots, labeled A through E, indicate the nodes of  $(\mu_j, \bar{q}_j)$  ( $j = 1, 2, \dots, M$ );  $M = 5$  in this example. Segments AB and CD are the steepest intervals in this example. The midpoints of the two steepest intervals are indicated by the plus symbols, labeled F and G. The gradients corresponding to cases (i), (ii), and (iii) are those of segments DE, FG, and GE, respectively. In this example, the last case gives the largest value, which is chosen as  $\beta^{(1)}$ . (b): Updating the profile. The original profile is the same as shown in (a), which is indicated by open circles connected by dashed lines. Since  $\beta^{(1)}$  is set to the gradient of segment GE, the gradient of the two steepest intervals (segments AB and CD) are reduced to  $\beta^{(1)}$  without moving the midpoints. Consequently, nodes A through D are moved to nodes A' through D', and a new profile (indicated by the solid lines) is formed. In this new profile, the steepest intervals are segment A'B' and the interval that consists of segments C'D' and D'E. Note that the gradient of segment B'C' is slightly smaller than that of the new steepest intervals.



ly. Ultimately, we can reach a profile in which all segments are included in only one steepest interval with a finite number of repetitions. This is because at least one more segment not included in the steepest intervals is incorporated into a steepest interval with further repetition. Representing the number of finite repetitions as  $n$ , we write the sequence of the obtained profiles as  $\bar{q}^{(0)}, \bar{q}^{(1)}, \dots, \bar{q}^{(n)}$ , and the corresponding values of maximum gradient and the total angular momentum as  $\beta^{(0)}, \beta^{(1)}, \dots, \beta^{(n)}$  and  $D^{(0)}, D^{(1)}, \dots, D^{(n)}$ , respectively. As stated above, the following inequalities hold:

$$\beta^{(0)} > \beta^{(1)} > \dots > \beta^{(n)} > 0; D^{(0)} > D^{(1)} > \dots > D^{(n)} > 0.$$

Furthermore, we can obtain a profile of constant zonal absolute vorticity from the profile  $\bar{q}^{(n)}$  through air parcel exchange using the procedure described above. The corresponding values of maximum gradient and the total angular momentum,  $\beta^{(n+1)}$  and  $D^{(n+1)}$ , respectively, are both zero.

Next, let us consider satisfying the condition of the conservation of the total angular momentum,  $D = D_{\text{initial}}$ . Since  $D^{(0)} > D_{\text{initial}} > 0 = D^{(n+1)}$ , we can find an integer  $k \in \{0, 1, \dots, n\}$  that satisfies  $D^{(k)} \geq D_{\text{initial}} > D^{(k+1)}$ . Then, we can choose  $\gamma (0 < \gamma \leq 1)$  to satisfy

$$\gamma D^{(k)} + (1 - \gamma) D^{(k+1)} = D_{\text{initial}},$$

which yields

$$\gamma = \frac{D_{\text{initial}} - D^{(k+1)}}{D^{(k)} - D^{(k+1)}}.$$

Using this  $\gamma$ , we can define a new profile as

$$\bar{q}^* = \gamma \bar{q}^{(k)} + (1 - \gamma) \bar{q}^{(k+1)}$$

so that the value of  $D$  for this new profile is equal to  $D_{\text{initial}}$ . The node indices included in the steepest intervals of  $\bar{q}^*$  are the same as those of  $\bar{q}^{(k)}$  and the value of the maximum gradient of  $\bar{q}^*$  is given as

$$\beta^* = \gamma \beta^{(k)} + (1 - \gamma) \beta^{(k+1)}$$

Note that the profile  $\bar{q}^*$  can be obtained from the profile  $\bar{q}^{(k)}$  through air parcel exchange at the steepest intervals (including endpoint nodes).

As explained above, we have completed the process of obtaining a profile  $\bar{q}^*$  that satisfies the conservation of the total angular momentum from the sorted profile of absolute vorticity through air parcel exchange only at the steepest intervals (including endpoint nodes).

#### 4.2 Proof

In this subsection, we prove that the profile  $\bar{q}^*$  constructed above gives the minimum of  $F_z$  under the constraints for  $(r_{ij})$  and the conservation of  $D$ . First, we

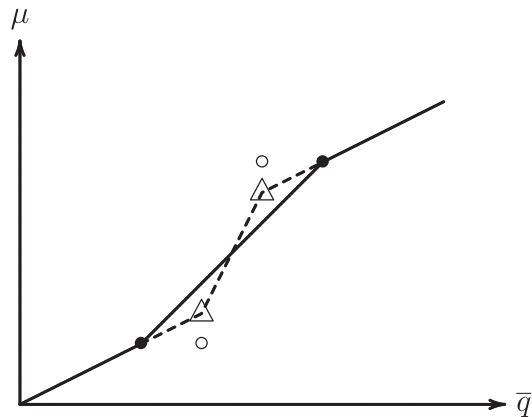


Fig. 2. Conceptual figure indicating the generation of the  $Q(\mu)$  profile. For the two endpoint nodes (indicated by the black dots) of the two steepest intervals, the corresponding nodes (each having the same  $\mu$  value as one of the endpoint nodes) of  $\bar{q}^{(0)}$  are indicated by open circles. The line of each steepest interval is extended to the  $\bar{q}$  value of the adjacent open circle and a new node (indicated by the triangle) is added. Connecting the new nodes yields the  $Q(\mu)$  profile (indicated by the dashed line).

determine the  $Q(\mu)$  profile that yields the revised Shepherd's bound based on the profile  $\bar{q}^*$ . Writing the components of  $\bar{q}^*$  as

$$\bar{q}^* = (\bar{q}_j^*)_{j=1,2,\dots,M},$$

we define  $Q(\mu)$  basically as a polyline that connects the points  $(\mu_j, \bar{q}_j^*)$  ( $j = 1, 2, \dots, M$ ). Just outside the steepest intervals (let us assume that  $(\mu_k, \bar{q}_k^*)$  is an endpoint of a steepest interval), however, we extend the line of the steepest interval until it crosses  $\bar{q} = \bar{q}_k^{(0)}$ , where we add a node and have  $Q(\mu)$  pass through this new node (Fig. 2).

With the above defined  $Q(\mu)$  and  $(r_{ij})$  that yields  $\bar{q}^*$  (note that  $\bar{q}_j = \bar{q}_j^*$  ( $j = 1, 2, \dots, M$ ) for this  $(r_{ij})$ ), we can show the following for (14).

- Since air parcel exchange from the profile  $\bar{q}^{(0)}$  occurs only at the steepest intervals,  $(q_i - \bar{q}_j) r_{ij} \neq 0$  only for  $\mu_i$ s that correspond to the steepest intervals.
- The range of the value of  $q$  in the steepest intervals is within  $\eta$ -intervals in which  $Y(\eta)$  has a gradient  $Y'_{\min} = 1/\beta^*$ . (We have defined  $Q(\mu)$  as described in Fig. 2 to satisfy this property.)

Therefore, in this case, the equality holds in (14).

Furthermore, since  $Q(\mu_j) = \bar{q}_j$  for  $\mu_j$  ( $j = 1, 2, \dots, M$ )

in this case, the equality holds in (15). Remembering that (15) gives an upper bound of  $F_w$ , the fact that we have obtained a case in which the equality holds indicates that  $\bar{q}^*$  and the corresponding  $(r_{ij})$  gives the maximum of  $F_w$  attainable by air parcel exchange that conserves the total angular momentum, and that the above defined  $Q(\mu)$  gives the minimum of the right-hand side of (15).

Thus, we have completed the proof for the equivalence of the revised Shepherd's bound and direct bound. We have also established a procedure to determine the profiles of  $Q(\mu)$  and  $\bar{q}^*$  that give the two bounds. While the two bounds are equivalent, the profile that connects  $(\mu_j, \bar{q}^*)$  ( $j = 1, 2, \dots, M$ ) by a polyline differs slightly from the profile of  $Q(\mu)$  because extrapolations are performed just outside the steepest intervals, as shown in Fig. 2. This difference, however, can be arbitrarily decreased by increasing the number of discretized belts,  $M$ .

## 5. Summary and discussion

We have demonstrated that the two upper bounds, direct bound and revised Shepherd's bound, introduced by Ishioka and Yoden (1996) for the growth of disturbances from barotropic instability are equivalent and that the corresponding profiles of  $\bar{q}^*$  and  $Q(\mu)$  coincide in the limit of very fine discretization. The proof is provided for a discretized system, and there may be mathematical difficulties in generalizing the procedure to obtain the profile  $\bar{q}^*$  in the continuous system. However, we believe that such a problem will have no essential difference between continuous and discretized systems in the limit of very fine discretization. The procedure to obtain  $\bar{q}^*$  introduced in the proof is a simple iteration algorithm that does not require an optimization method. Furthermore, the number of required iterations for the procedure is less than the number of discretizations,  $M$ . Also, very little memory is required because the procedure does not deal with  $(r_{ij})$  but only updates  $\bar{q}$ . While the spherical geometry has been considered in this study, the proof is applicable to the channel geometry with slip boundary condition.

Before concluding, let us consider why the proposed procedure yields the minimum of  $F_z$  (or the maximum of  $F_w$ ). In the proposed procedure, air parcel exchange occurs at latitudes where  $(\bar{q}_{j+1} - \bar{q}_j)/\Delta\mu$  has the maximum value. The reason for this is explained as follows. Let us assume that in two belts  $\mu_a$  and  $\mu_b$  ( $\mu_a < \mu_b$ ), zonal mean values of absolute vorticity are  $\bar{q}_a$  and  $\bar{q}_b$ , respectively ( $\bar{q}_a < \bar{q}_b$ ), and that air parcel exchange between these two belts generates zonal mean values

of absolute vorticity  $\bar{q}_a + \delta q$  and  $\bar{q}_b - \delta q$  (at  $\mu_a$  and  $\mu_b$ , respectively). Here we assume that  $\delta q > 0$ . The increment of  $D$  by air parcel exchange is written as

$$\delta D = -\frac{1}{M}(\mu_b - \mu_a)\delta q < 0,$$

and the increment of  $F_z$  is written as

$$\begin{aligned} \delta F_z &= \frac{1}{2M}\{(\bar{q}_a + \delta q)^2 + (\bar{q}_b - \delta q)^2 - \bar{q}_a^2 - \bar{q}_b^2\} \\ &= \frac{1}{M}\{-(\bar{q}_b - \bar{q}_a)\delta q + (\delta q)^2\} \\ &\approx -\frac{1}{M}(\bar{q}_b - \bar{q}_a)\delta q < 0. \end{aligned}$$

Here in the last simplification, we neglected the quadratic term of  $\delta q$ , which we assume to be sufficiently small. Finally, we obtain

$$\frac{\delta F_z}{\delta D} \approx \frac{\bar{q}_b - \bar{q}_a}{\mu_b - \mu_a}. \quad (16)$$

Since the right-hand side of (16) is the gradient of the line connecting two points  $(\mu_a, \bar{q}_a)$  and  $(\mu_b, \bar{q}_b)$ , (16) implies that air parcel exchange should occur in intervals where the gradient of zonal mean absolute vorticity is the steepest to maximize the decrease of  $F_z$  per the decay of  $D$ . Therefore, the procedure introduced in Section 4 can be interpreted as follows. First, the value of  $D$  is maximized by sorting the initial distribution of absolute vorticity. Second, air parcel exchange occurs from intervals where the gradient of zonal mean absolute vorticity is the steepest. Finally, the profile can be reached that gives the minimum of  $F_z$  under the constraint of having the same value of  $D$  as the initial profile.

The above interpretation of the procedure to obtain  $\bar{q}^*$  seems analogous to annealing, which, in metallurgy, refers to the process of changing the crystal structure of metal to a lower free energy state by increasing and then gradually decreasing the temperature of the metal. If we replace the total angular momentum  $D$  and the zonal enstrophy  $F_z$  with the temperature and free energy, respectively, the analogy will be clear. This analogy also reminds us that we cannot always assume that a path on which  $\delta F_z/\delta D$  is the largest leads to the true minimum of  $F_z$ . Hence, the above interpretation alone is not sufficient for a proof. To complete a proof in this manner, it will be necessary to examine the geometric structure of the isosurfaces of  $D$  and  $F_z$  in the multidimensional space of  $(r_{ij})$  under the constraints. On the other hand, the proof given in Section 4 clearly shows that the iterative algorithm leads to the true minimum of  $F_z$  using the revised



Shepherd's bound.

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